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On Uniqueness of L_1-Approximation for Certain Families of Spline Functions

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In a recent note Strauss [3] proved uniqueness of the best L_1-approximation when approximating with spline functions. In this note we consider spline functions for which only continuity (and no differentiability) is required in the knots. Then the underlying subspaces need only satisfy the Haar condition; the existence of a Markov chain is not required. Moreover, the idea of our proof may easily be transferred to the case considered by Strauss in order to obtain a shorter proof of the uniqueness theorem. On the other hand no characterization is established.

Let $a = t_0 < t_1 < t_2 < \cdots < t_k = b$ be a decomposition of $I = [a, b]$ into k intervals. Let $V_{n,k}$ denote the subspace of $C[a, b]$ of those functions, g, such that on any subinterval $[t_{i-1}, t_i]$, $i = 1, 2,..., k$, g belongs to a given n-dimensional Haar-subspace, $n \geqslant 2$.

To prove the uniqueness of L_1-approximation we need the following lemmas:

LEMMA 1. *If $f \in C[\alpha, \beta]$ and if g_1, g_2 are two best L_1-approximations in a convex set $Y \subset C(X)$, then $g_1 - g_2$ vanishes at the zeros of $f - \frac{1}{2}(g_1 + g_2)$.*

This statement was the first step of Cheney's proof of Jackson's Theorem [1].

LEMMA 2. *Assume that m distinct points $z_1, z_2,..., z_m$ in $[\alpha, \beta]$ are given, and $m \leqslant n - 1$. Then in an n-dimensional Haar-subspace there is a function*

$h \not\equiv 0$, whose zeros in $[\alpha, \beta]$ are precisely $z_1, z_2, ..., z_m$. Moreover, $h(t)$ changes its sign at the zeros in (α, β).

In slightly different terms this can be found in Karlin and Studden [2].

Now we are ready to prove our main theorem.

UNIQUENESS THEOREM. *For every $f \in C[a, b]$ there is a unique best L_1-approximation in $V_{n,k}$.*

Proof. Assume that g_1 and g_2 are two best approximations. Call $[t_{i-1}, t_i]$, $i = 1, 2, ..., k$, a Z-interval (an NZ-interval, respectively) if $g_1 - g_2$ vanishes identically (does not vanish identically, respectively) on $[t_{i-1}, t_i]$. Assume there is at least one NZ-interval. Let $[t_l, t_m]$, $0 \leqslant l < m \leqslant k$ be a maximal block of NZ-intervals, i.e., $[t_l, t_m]$ contains only NZ-intervals, but $[t_{l-1}, t_m]$ and $[t_l, t_{m+1}]$ either are not well-defined or don't have this property.

Put $g = (g_1 + g_2)/2$. We will construct a function $h \in V_{n,k}$, $h \not\equiv 0$, satisfying

$$h(t)(f - g)(t) \geqslant 0, \qquad t \in I \tag{1}$$

$$h(t) = 0, \qquad t \in I \backslash [t_l, t_m]. \tag{2}$$

Consider the interval $[t_j, t_{j+1}]$, $j = l, l + 1, ..., m - 1$. There, $g_1 - g_2$ has at most $n - 1$ zeros. Pick out those zeros, which are either endpoints of the interval or at which $f - g$ changes its sign. By Lemma 2 in the restricted family $V_{n,k} \mid [t_j, t_{j+1}]$ there is a function h_j, $h_j \neq 0$, which vanishes at exactly the points considered. After multiplying h_j with (-1) if necessary, we have

$$h_j(t)(f - g)(t) \geqslant 0, \qquad t \in [t_j, t_{j+1}]. \tag{3}$$

Now h is defined recursively. At first let h coincide with h_l on $[t_l, t_{l+1}]$. Assume that h has already been defined on $[t_l, t_j]$. If $j < m$ we consider two cases.

Case 1. $g_1(t_j) = g_2(t_j)$. Then we have $h(t_j) = h_{j-1}(t_j) = h_j(t_j) = 0$, and the domain of h can be extended by setting

$$h(t) = h_j(t), \qquad t \in [t_j, t_{j+1}]. \tag{4}$$

Case 2. $g_1(t_j) \neq g_2(t_j)$. Then it follows from Lemma 1 that $(f - g)(t_j) \neq 0$. Hence

$$\operatorname{sign} h(t_j) = \operatorname{sign}(f - g)(t_j) = \operatorname{sign} h_j(t_j) \neq 0.$$

After multiplying h_j with an appropriate positive factor if necessary, we have $h_j(t_j) = h(t_j)$ and the extension of the domain may also be performed by (4).

Observe that $g_1 - g_2$ vanishes at t_l if $l > 0$ and vanishes at t_m if $m < k$. Hence, by (2) h is defined as a continuous function, and $h \in V_{n,k}$.

Setting $\epsilon(t) = f(t) - g(t)$, from (1) we obtain $\int_{t_l}^{t_m} h(t) \operatorname{sign} \epsilon(t)\, dt > 0$. Since $\epsilon(t)$ has at most a finite number of zeros in $[t_l, t_m]$, applying Lemma 1 of Cheney's paper to $[x_l, x_m]$, we obtain for sufficiently small λ

$$\int_I | \epsilon - \lambda h |\, dt = \int_{t_l}^{t_m} | \epsilon - \lambda h |\, dt + \int_{I \setminus [t_l, t_m]} | \epsilon |\, dt$$

$$< \int_{t_l}^{t_m} | \epsilon |\, dt + \int_{I \setminus [t_l, t_m]} | \epsilon |\, dt = \int_I | \epsilon |\, dt,$$

a contradiction to optimality.

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References

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